

Embedding Crossed Products into a Unital Simple AF-algebra

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Abstract

Let X be a compact metric space and let α be a homeomorphism on X . Related to a theorem of Pimsner, we show that $C(X) \rtimes_{\alpha} \mathbb{Z}$ can be embedded into a unital simple AF-algebra if and only if there is a strictly positive α -invariant Borel probability measure.

Suppose that Λ is a \mathbb{Z}^d action on X . If $C(X) \rtimes_{\Lambda} \mathbb{Z}$ can be embedded into a unital simple AF-algebra, then there must exist a strictly positive Λ -invariant Borel probability measure. We show that, if in addition, there is a generator α_1 of Λ such that (X, α_1) is minimal and unique ergodic, then $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$ can be embedded into a unital simple AF-algebra with a unique tracial state.

Let A be a unital separable amenable simple C^* -algebra with tracial rank zero and with a unique tracial state which satisfies the Universal Coefficient Theorem and let G be a finitely generated discrete abelian group. Suppose $\Lambda : G \rightarrow \text{Aut}(A)$ is a homomorphism. Then $A \rtimes_{\Lambda} G$ can always be embedded into a unital simple AF-algebra.

1 Introduction

Let X be a compact metric space and let α be a homeomorphism on X . It was proved by Pimsner ([16]) that $C(X) \rtimes_{\alpha} \mathbb{Z}$ can be embedded into an AF-algebra if and only if α is pseudo-non-wondering. Let A be a unital AF-algebra and let $\alpha \in \text{Aut}(A)$. Nate Brown proved the following AF-embedding theorem: $A \rtimes_{\alpha} \mathbb{Z}$ can be embedded into an AF-algebra if and only if $A \rtimes_{\alpha} \mathbb{Z}$ is quasidiagonal. He also gave a K -theoretical necessary and sufficient condition for which $A \rtimes_{\alpha} \mathbb{Z}$ can be embedded into an AF-algebra. This result has since been improved, at least partially, by Matui ([13]). For non- \mathbb{Z} actions, Dan Voiculescu asked when $C(X) \rtimes_{\Lambda} \mathbb{Z}^2$ can be embedded into an AF-algebra.

An AF-algebra may have some infinite feature. Let \mathcal{K} be the C^* -algebra of compact operators on ℓ^2 . Then an essential extension $0 \rightarrow \mathcal{K} \rightarrow E \rightarrow M_n \rightarrow 0$, where M_n is a matrix algebra, gives a unital AF-algebra. This AF-algebra E does not have a faithful tracial state. In particular, E can not be embedded into a unital simple AF-algebra. For a unital stably finite C^* -algebra A with a faithful tracial state, a stronger and, perhaps, a more interesting embedding question is when A can be embedded into a unital simple AF-algebra. In this note, we will present some progress on the last question.

We consider the following question: When can $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$ be embedded into a unital simple AF-algebra? An obvious necessary condition for $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$ to be embedded into a unital simple AF-algebra is that there is a strictly positive Λ -invariant Borel probability measure on X (see 2.2 below). We show that when $d = 1$, if there is such a measure, then, indeed, $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$ can be embedded into a unital simple AF-algebra. When $d > 1$, if in addition, there is a generator α_1 of action Λ such that (X, α_1) is minimal and unique ergodic, we show that $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$ can be embedded into a unital simple AF-algebra.

Turn to non-commutative cases, it was shown by N. Brown ([3]) that, among other things, if A is a UHF-algebra, then $A \rtimes_{\Lambda} G$ can be embedded into an AF-algebra, where G is a finitely generated discrete abelian group and where $\Lambda : \mathbb{Z}^d \rightarrow \text{Aut}(A)$ is a homomorphism. Let A be a unital separable amenable simple C^* -algebra with tracial rank zero and with a unique tracial

state which satisfies the UCT and let $\Lambda : G \rightarrow \text{Aut}(A)$. We show $A \rtimes_{\Lambda} G$ can be embedded into a unital simple AF-algebra. In particular, if A is a unital simple AT-algebra with a unique tracial state, $A \rtimes_{\Lambda} G$ can be embedded into a unital simple AF-algebra.

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2 Preliminaries

2.1. Let A be a C^* -algebra. Denote by $\text{Aut}(A)$ the group of automorphisms on A .

Let A be a stably finite C^* -algebra. Denote by $T(A)$ the tracial state space of A and by $\text{Aff}(T(A))$ the normed space of all real affine continuous functions on $T(A)$. Denote by $\rho_A : K_0(A) \rightarrow \text{Aff}(T(A))$ the positive homomorphism induced by $\rho_A([e]) = (\tau \otimes T_k)(e)$, where e is a projection in $A \otimes M_k$ and T_k is the standard trace on M_k , $k = 1, 2, \dots$

All ideals in this paper are closed two-sided ideals.

Definition 2.2. Let X be a compact metric space and let $\alpha_1, \alpha_2, \dots, \alpha_d$ be homeomorphisms on X , where $d \geq 1$ is an integer. Denote by $\alpha_j^* : C(X) \rightarrow C(X)$ the automorphism defined by $\alpha_j^*(f) = f \circ \alpha_j$ for all $f \in C(X)$, $j = 1, 2, \dots, d$. Suppose that $\alpha_j \circ \alpha_i = \alpha_i \circ \alpha_j$, $i, j = 1, 2, \dots, d$. Then it gives a \mathbb{Z}^d action on X . This gives a homomorphism, $\Lambda : \mathbb{Z}^d \rightarrow \text{Aut}(C(X))$. The crossed product will be denoted by $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$. If $d = 1$, we also use $C(X) \rtimes_{\alpha_1} \mathbb{Z}$ for the crossed product.

A measure μ on X is said to be *strictly positive Λ -invariant* measure, if $\mu(O) > 0$ for any non-empty open subset $O \subset X$ and $\mu(\alpha_j(E)) = \mu(E)$ for any Borel subset $E \subset X$, $j = 1, 2, \dots, d$.

Definition 2.3. Let A be a unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. Denote by $A \rtimes_{\alpha} \mathbb{Z}$ the crossed product. In this paper, we will fix a unitary and denote it by z_{α} for which $\text{ad } z_{\alpha}(a) = \alpha(a)$ for all $a \in A$.

2.4. Let X be a metric space and let $x \in X$. For $\delta > 0$, we will use $O_{\delta}(x)$ for the open ball with center at x and with radius δ .

For any subset $S \subset X$, \overline{S} is the closure of S . we denote by $\partial(S)$ the boundary $\overline{S} \setminus S$ of S .

Definition 2.5. Let X be a compact metric space and let $d > 0$. A subset $S \subset X$ is said to be *d-connected* if for any two points $x, y \in S$, there are $x_0, x_1, \dots, x_n \in S$ such that

$$\text{dist}(x_i, x_{i+1}) < d \text{ and } x = x_0, y = x_n$$

$i = 0, 1, \dots, n-1$. S is said to be a *d-connected* component if S is a closed and open subset of X which is *d-connected*. Clear that every connected component of X is *d-connected* component of X .

Definition 2.6. Denote by \mathcal{U} throughout this paper the universal UHF-algebra $\mathcal{U} = \otimes_{n \geq 1} M_n$.

Let $\{e_{i,j}^{(n)}\}$ be the canonical matrix units for M_n . Let $u_n \in M_n$ be the unitary matrix such that $\text{ad } u_n(e_{i,i}^{(n)}) = e_{i+1,i+1}^{(n)}$ (modulo n). Let $\sigma = \otimes_{n \geq 1} \text{ad } u_n \in \text{Aut}(\mathcal{U})$ be the shift (see for example Example 2.2 of [1]). A fact that we will use in this paper is the following cyclic Rokhlin

property that σ has: For any integer $k > 0$, any $\epsilon > 0$ any finite subset $\mathcal{F} \subset \mathcal{U}$, there exist mutually orthogonal projections $e_1, e_2, \dots, e_k \in \mathcal{U}$ such that

$$\sum_{i=1}^k e_i = 1_{\mathcal{U}}, \quad (\text{e2.1})$$

$$\|xe_i - e_i x\| < \epsilon \text{ for all } x \in \mathcal{F} \text{ and} \quad (\text{e2.2})$$

$$\sigma(e_i) = e_{i+1}, i = 1, 2, \dots, k \text{ } (e_{k+1} = e_1). \quad (\text{e2.3})$$

We will frequently use the following result.

Theorem 2.7. (Theorem 3.4 of [12])

Let A be a unital separable simple C^ -algebra with tracial rank zero and let $\alpha \in \text{Aut}(A)$. Suppose that α satisfies the tracial cyclic Rokhlin property. Suppose also that there is a subgroup $G \subset K_0(A)$ for which $\rho_A(K_0(A))$ is dense in $\text{Aff}(T(A))$ such that $(\alpha^r)_{*0}|_G = \text{id}_G$ for some integer $r \geq 1$. Then $A \rtimes_{\alpha} \mathbb{Z}$ has tracial rank zero.*

If, in addition, A is assumed to be amenable and satisfy the Universal Coefficient Theorem, $A \rtimes_{\alpha} \mathbb{Z}$ is a unital simple AH-algebra with real rank zero and with no dimension growth, by the classification theorem of [9].

3 Embedding into the Cantor systems

Lemma 3.1. *Let X be a compact metric space and μ be a Borel probability measure on X . Then, for any $x \in X$ and any $\delta > 0$, there exists $0 < r < \delta$ such that*

$$\mu(\partial(O_r(x))) = 0.$$

This is known. See the proof of 3.2 of [11] for example.

Lemma 3.2. *Let X be an infinite compact metric space and $\Lambda : \mathbb{Z}^d \rightarrow \text{Aut}(C(X))$ be a homomorphism. Suppose that there is a Λ -invariant strictly positive Borel probability measure μ . Then, there is a unital simple AF-algebra C_0 with a unique trace τ and a unital abelian C^* -subalgebra $C_{00} \subset C_0$ which have the following properties:*

- (1) $\rho_{C_0}(K_0(A)) = K_0(A) = \mathbb{D}$, where \mathbb{D} is a countable divisible dense subgroup of \mathbb{R} ,
- (2) there exists a monomorphism $h : C(X) \rightarrow C_{00}$ such that

$$\tau \circ h(f) = \int_X f d\mu \text{ for all } f \in C(X), \quad (\text{e3.4})$$

- (3) for each $j \in \{1, 2, \dots, d\}$, there exists path of unitaries $\{u_{j,t} : t \in [1, \infty)\}$ in C_0 such that

$$\lim_{t \rightarrow \infty} \|h(f \circ \alpha_j) - \text{ad } u_{j,t} \circ h(f)\| = 0 \quad (\text{e3.5})$$

for all $f \in C(X)$ and

- (4) C_{00} is an AF-algebra with $1_{C_{00}} = 1_{C_0}$ and there exists a sequence of mutually commuting projections $\{e_{n,i}\}$ which generates C_{00} . Moreover

$$u_{j,n+t}^* e_{n,i} u_{j,n+t} = u_{j,n}^* e_{n,i} u_{j,n} \text{ and} \quad (\text{e3.6})$$

$$u_{j',n+1+t}^* u_{j,n+t}^* e_{n,i} u_{j,n+t} u_{j',t+1} = u_{j,t}^* u_{j',n+1+t}^* e_{n,i} u_{j',n+1+t} u_{j,n+t} \quad (\text{e3.7})$$

for all n, i, j and $t \geq 0$.

Proof. Fix a decreasing sequence of positive numbers $\{d_n\}$ for which $\lim_{n \rightarrow \infty} d_n = 0$.

Let $X_{n,1}, X_{n,2}, \dots, X_{n,m(n)}$ be disjoint d_n -connected components with $\cup_i X_{n,i} = X$. There are finitely many open subsets $O_1, \dots, O_{m'(n)}$ such that

$$\cup_{m=1}^{m'(n)} O_m = X \text{ and } \mu(\partial(O_j)) = 0, j = 1, 2, \dots, m'(n).$$

From this, it is easy to obtain mutually disjoint Borel subsets $Y'_{n,1}, Y'_{n,2}, \dots, Y'_{n,l'(n)}$ for which each $Y'_{n,l'}$ ($l' = 1, 2, \dots, l'(n)$) has diameter less than d_n and $\cup_{l=1}^{l'(n)} Y'_{n,l} = X$. Moreover, each $Y'_{n,l'}$ has the form $O \cup S$, where O is an open subset and $S \subset \partial(O)$ is a Borel subset and $\mu(\partial(Y'_{n,l'})) = 0$.

Furthermore, we assume that $\{Y'_{n,l'} : l' = 1, 2, \dots, l'(n)\}$ is a refinement of $\{X_{n,1}, X_{n,2}, \dots, X_{n,m(n)}\}$. Let

$$Y_{n,1}, Y_{n,2}, \dots, Y_{n,l(n)}$$

be mutually disjoint Borel subsets for which each $Y_{n,l}$ ($l = 1, 2, \dots, l(n)$) has diameter less than d_n and the collection of finite union of these $Y_{n,l}$'s contains

$$\{X_{n,i}, \alpha_j^{-1}(X_{n,i}) : i = 1, 2, \dots, m(n)\} \cup \{Y'_{n,l}, \alpha_j^{-1}(Y'_{n,l}) : l = 1, 2, \dots, l(n)\}.$$

$j = 1, 2, \dots, d$, $n = 1, 2, \dots$. Moreover, We may assume that we also assume that each $Y_{n,l}$ has the form $O \cup S$, where O is an open and $S \subset \partial(O)$ is a Borel subset as well as $\mu(\partial(Y_{n,l})) = 0$.

We may assume that

$$Y'_{n+1,1}, Y'_{n+1,2}, \dots, Y'_{n+1,l'(n+1)}$$

is a refinement of $\{Y_{n,1}, Y_{n,2}, \dots, Y_{n,l(n)}\}$. By induction, it follows that the partition $\{Y_{n+1,1}, Y_{n+1,2}, \dots, Y_{n+1,l(n+1)}\}$ is a refinement of $\{Y_{n,1}, Y_{n,2}, \dots, Y_{n,l(n)}\}$, $n = 1, 2, \dots$. Define \mathbb{D} to be a countable divisible group of \mathbb{R} which contains \mathbb{Q} and $\{\mu(Y_{n,i}) : i = 1, 2, \dots, l(n), n = 1, 2, \dots\}$. Let C_0 be a unital separable simple AF-algebra with unique trace τ such that $(K_0(C_0), K_0(C_0)_+, [1_{C_0}]) = (\mathbb{D}, \mathbb{D}_+, 1)$.

There are mutually orthogonal projections $\{e_{n,i} : i = 1, 2, \dots, l(n)\}$ such that $\tau(e_{n,i}) = \mu(Y_{n,i})$, $i = 1, 2, \dots, l(n)$. It follows that $\sum_i e_{n,i} = 1_{C_0}$. For a fixed n , if $Y_{n+1,j} \subset Y_{n,i}$, one can obtain projections $\{e_{n+1,j}\}$ in the C^* -subalgebra $e_{n,i}C_0e_{n,i}$. Moreover, if for some finite subset J , $\cup_{j \in J} Y_{n+1,j} = Y_{n,i}$, then $\sum_{j \in J} e_{n+1,j} = e_{n,i}$. Therefore, we may assume that $\{e_{n,i} : i = 1, 2, \dots, l(n), n = 1, 2, \dots\}$ is also a set of pairwise commuting projections. Let C_{00} be the commutative C^* -subalgebra of C_0 generated by $\{e_{n,i} : i = 1, 2, \dots, l(n), n = 1, 2, \dots\}$. Define $h_n : C(X) \rightarrow C_{00}$ by

$$h_n(f) = \sum_{i=1}^{l(n)} f(x_{n,i}) e_{n,i} \text{ for all } f \in C(X), \quad (\text{e3.8})$$

where $x_{n,i} \in Y_{n,i}$, $i = 1, 2, \dots, l(n)$ and $n = 1, 2, \dots$. For any $m > n$ and $i \in \{1, 2, \dots, l(n)\}$, there exists $J(i) \subset \{1, 2, \dots, l(m)\}$ such that $\sum_{j \in J(i)} e_{m,j} = e_{n,i}$. We have

$$h_n(f) - h_m(f) = \sum_{i=1}^{l(n)} \sum_{j \in J(i)} (f(x_{n,i}) - f(x_{m,j})) e_{m,j} \text{ for all } f \in C(X). \quad (\text{e3.9})$$

Note that $x_{n,i}, x_{m,j} \in Y_{n,i}$. Since $\lim_{n \rightarrow \infty} d_n = 0$, we conclude that, for each $f \in C(X)$, $\{h_n(f)\}$ is Cauchy in C_{00} . Define $h(f) = \lim_{n \rightarrow \infty} h_n(f)$ for all $f \in C(X)$.

Write $C_{00} = C(Y)$, where Y is a compact totally disconnected metric space. Let $\Omega_{n,i}$ be the clopen subset corresponding to $e_{n,i}$. Let $f_1 \in C(X)$ such that $f_1(x) = 0$ for all $x \notin Y_{n,i}^o$, the interior of $Y_{n,i}$ and let $f_2 \in C(X)$ such that $f_2(x) = 1$ for all $x \in \overline{Y_{n,i}}$. It follows from (e3.8) and (e3.9) that

$$h(f_1)e_{n,i} = h(f_1) \text{ and } h(f_2)e_{n,i} = e_{n,i} \quad (\text{e3.10})$$

which will be used later.

Fix $n \geq 2$. For each j and i , there is a finite subset $J(i, n, j) \subset \{1, 2, \dots, l(n+1)\}$ and a projection with the form $\sum_{i' \in J(i, n, j)} e_{n+1, i'}$ which corresponds to $\alpha_j^{-1}(Y_{n, i})$. Denote this projection by $p_{n, i, j}$. There is $\xi_{n, i} \in \alpha_j^{-1}(Y_{n, i})$ such that $\alpha_j(\xi_{n, i}) = x_{n, i}$. Since $\lim_{n \rightarrow \infty} d_n = 0$, we have

$$\lim_{n \rightarrow \infty} \left\| \sum_{k=1}^{l(n+1)} f(\alpha_j(x_{n+1, i})) e_{n+1, k} - \sum_{i=1}^{l(n)} f(x_{n, i}) p_{n, i, j} \right\| = 0 \quad (\text{e 3.11})$$

for all $f \in C(X)$. Fix j and fix $n \geq 2$, there is a unitary $u_{j, n} \in C_0$ such that

$$u_{j, n}^* e_{n, i} u_{j, n} = p_{n, i, j}, \quad j = 1, 2, \dots, d \quad (\text{e 3.12})$$

$i = 1, 2, \dots, l(n)$.

Suppose that $u_{j, n}$ has been defined. Suppose that $S(i, n) \subset \{1, 2, \dots, l(n+1)\}$ such that $\sum_{k \in S(i, n)} \alpha_j^{-1}(Y_{n+1, k}) = \alpha_j^{-1}(Y_{n, i})$. Then, for $k \in S(i, n)$,

$$u_{j, n}^* e_{n+1, k} u_{j, n} \leq p_{n, i, j} \quad \text{and} \quad \sum_{k \in S(i, n)} u_{j, n}^* e_{n+1, k} u_{j, n} = p_{n, i, j}. \quad (\text{e 3.13})$$

Let $J(k, n+1) \subset \{1, 2, \dots, l(n+2)\}$ so that $\cup_{i' \in J(k, n+1)} Y_{n+2, i'} = \alpha_j^{-1}(Y_{n+1, k})$. Let

$$p_{n+1, k, j} = \sum_{i' \in J(k, n+1)} e_{n+2, i'}. \quad (\text{e 3.14})$$

We have

$$\tau(u_{j, n}^* e_{n+1, k} u_{j, n}) = \tau(p_{n+1, k, j}). \quad (\text{e 3.15})$$

Therefore there is a partial isometry $w(j, n+1, i, k) \in p_{n, i, j} C_0 p_{n, i, j}$ such that ($k \in S(i, n)$)

$$\begin{aligned} w(j, n+1, i, k)^* w(j, n+1, i, k) &= p_{n+1, k, j} \quad \text{and} \\ w(j, n+1, i, k) w(j, n+1, i, k)^* &= u_{j, n}^* e_{n+1, k} u_{j, n}. \end{aligned} \quad (\text{e 3.16})$$

Put $w(j, n+1, i) = \sum_{k \in S(i, n)} w(j, n+1, i, k)$. Since

$$\sum_{k \in S(i, n)} p_{n+1, k, j} = p_{n, i, j}, \quad (\text{e 3.17})$$

$w(j, n+1, i)$ is a unitary in $p_{n, i, j} C_0 p_{n, i, j}$. There is a path of unitaries in $\{W(j, n+1, t, i) : t \in [0, 1]\}$ in $p_{n, i, j} C_0 p_{n, i, j}$ with length no more than π for which

$$W(j, n+1, i, 0) = p_{n, i, j} \quad \text{and} \quad W(j, n+1, i, 1) = w(j, n+1, i). \quad (\text{e 3.18})$$

Define

$$U_{j, n}(t) = \sum_{i=1}^{l(n)} W(j, n+1, i, t) u_{j, n} \quad \text{and} \quad u_{j, n+1} = \sum_{i=1}^{l(n)} W(j, n+1, i, 1) u_{j, n}. \quad (\text{e 3.19})$$

Then

$$U_{j, n}(0) = u_{j, n}, \quad U_{j, n}(1) = u_{j, n+1}, \quad (\text{e 3.20})$$

$$u_{j, n+1}^* e_{n+1, k} u_{j, n+1} = p_{n+1, k, j} \quad \text{and} \quad U_{j, n}(t)^* e_{n, i} U_{j, n}(t) = p_{n, i, j} \quad (\text{e 3.21})$$

Define $u_{j,s} = U_{j,n}(s - n)$ for $s \in [n, n + 1)$. Thus, for any n and i ,

$$u_{j,s}^* e_{n,i} u_{j,s} = u_{j,n}^* e_{n,i} u_{j,n} \quad (\text{e 3.22})$$

for any $s \geq n$.

Suppose that $J'(i, n + 1, j, j_1) \subset \{1, 2, \dots, l(n + 2)\}$ such that $\sum_{i_1 \in J'(i, n + 1, j, j_1)} e_{n+2, i_1}$ corresponds to $\alpha_{j_1}^{-1} \circ \alpha_j^{-1}(Y_{n,i})$. Then since $\alpha_{j'}^{-1} \circ \alpha_j^{-1}(Y_{n,i}) = \alpha_j^{-1} \circ \alpha_{j'}^{-1}(Y_{n,i})$,

$$\sum_{i_1 \in J'(i, n + 1, j, j_1)} e_{n+2, i_1} = \sum_{i_2 \in J'(i, n + 1, j_1, j)} e_{n+2, i_2} \quad (\text{e 3.23})$$

This implies that

$$u_{j_1, n+1}^* u_{j,n}^* e_{n,i} u_{j,n} u_{j_1, n+1} = u_{j,n}^* u_{j_1, n+1}^* e_{n,i} u_{j_1, n+1} u_{j,n}. \quad (\text{e 3.24})$$

By applying (e 3.11), we obtain that

$$\lim_{t \rightarrow \infty} \|h(f \circ \alpha_j) - \text{ad} u_{j,t} \circ h(f)\| = 0 \quad (\text{e 3.25})$$

for all $f \in C(X)$ and $j = 1, 2, \dots, d$. We also have

$$\tau \circ h(f) = \int_X f d\mu \quad (\text{e 3.26})$$

for all $f \in C(X)$. From (e 3.24) and (e 3.21), one also has both (e 3.6) and (e 3.7). \square

Corollary 3.3. *Let X be a compact metric space and let $\Lambda : \mathbb{Z}^d \rightarrow \text{Aut}(C(X))$ be a homomorphism. Suppose that there is a strictly positive Λ -invariant Borel probability measure μ . Then there is an embedding $h_1 : C(X) \rightarrow C(Y)$, where Y is a compact totally disconnected metric space and there is an embedding $h_2 : C(Y) \rightarrow C_0$, where C_0 is a unital simple AF-algebra with a unique tracial state τ satisfying the following:*

(1) *For each $j = 1, 2, \dots, d$, there is a path of unitaries $\{u_{j,t} : t \in [1, \infty)\}$ in C_0 such that*

$$\lim_{t \rightarrow \infty} \|\text{ad} u_{j,t} \circ h_2 \circ h_1(f) - \alpha_j^* \circ h_1(f)\| = 0 \quad (\text{e 3.27})$$

for all $f \in C(X)$.

(2)

$$\lim_{t \rightarrow \infty} \text{ad} u_{j,t}(h_2(g)) = \beta_j^*(g) \quad (\text{e 3.28})$$

defines an automorphism on $C(Y)$.

Moreover $\beta_1^, \beta_2^*, \dots, \beta_d^*$ defines a \mathbb{Z}^d action $\overline{\Lambda}$ on $C(Y)$ and*

$$\beta_j^* \circ h_1 = h_1 \circ \alpha_j^*, \quad j = 1, 2, \dots, d.$$

Proof. We will use the notation in the proof of 3.2. Since C_0 is a unital separable commutative AF-algebra, there is a totally disconnected compact metric space Y such that $C_{00} \cong C(Y)$. Note that (1) directly follows from 3.2. To see (2), we note that, for each projection $e \in C_{00}$, as in the proof of 3.2, by (e 3.22), there is $t_0 \geq 1$ such that

$$u_{j,t}^* e u_{j,t} = u_{j,s}^* e u_{j,s} \quad (\text{e 3.29})$$

for all $t, s \geq t_0$ and $j = 1, 2, \dots, d$. In particular, $u_{j,t}^* e u_{j,t}$ are in C_{00} for $t \geq t_0$. It follows that $\{u_{j,t}^* g u_{j,t}\}$ converges to an element in C_{00} . Thus $\lim_{t \rightarrow \infty} \text{ad } u_{j,t}(h_2(g)) = \beta_j^*(g)$ defines an automorphism on $C(Y)$. It follows from (4) of 3.2 that $\beta_1^*, \beta_2^*, \dots, \beta_d^*$ generate a \mathbb{Z}^d action $\bar{\Lambda}$ on $C(Y)$. Since

$$\beta_j^*(h_1(f)) = \lim_{t \rightarrow \infty} \text{ad } u_{j,t}(h_2 \circ h_1(f)) = h_1 \circ \alpha_j^*(f) \quad (\text{e 3.30})$$

for all $f \in C(X)$, $\beta_j \circ h_1 = h_1 \circ \alpha_j^*$, $j = 1, 2, \dots, d$. So the last part of the corollary follows. \square

Corollary 3.4. *Let X be a compact metric space and let $\Lambda : \mathbb{Z}^d \rightarrow \text{Aut}(C(X))$ be a homomorphism. Suppose that there is a strictly positive Λ -invariant Borel probability measure μ . Let $C_0, C(Y) \cong C_{00}$, $h_1 : C(X) \rightarrow C(Y)$ and $h_2 : C(Y) \rightarrow C_0$ be as in 3.3 and as constructed in 3.2. Denote by $\beta_j : Y \rightarrow Y$ the homeomorphism induced by the automorphism β_j^* . Suppose that α_1 is minimal. Then $\beta_1 : Y \rightarrow Y$ is also minimal and Y is homeomorphic to the Cantor set, if X is infinite. If (X, α_1) has a unique α_1 -invariant Borel probability measure, then (Y, β_1) has a unique β_1 -invariant Borel probability measure.*

Proof. Fix a continuous surjective map $s : Y \rightarrow X$ for which $h_1(f)(y) = f \circ s(y)$ for all $y \in Y$. Fix $\xi \in Y$ and let $\zeta = s(\xi)$. We will keep notations used in the proof of 3.2.

First we claim the following: If $\delta > 0$ and if $\overline{Y_{n,i}} \cap O_\delta(\zeta) = \emptyset$, then there is no $y \in \Omega_{n,i}$ such that $s(y) = \zeta$, where $\Omega_{n,i} \subset Y$ is the clopen set associated with $e_{n,i}$.

Let $O_1, O_2 \subset X$ be open subsets containing $\overline{Y_{n,i}}$ such that $\overline{O_1} \subset O_2$ and $O_2 \cap O_{\delta/2}(\zeta) = \emptyset$. Let $f_1 \in C(X)$ such that $0 \leq f_1(x) \leq 1$ for all $x \in X$, $f_1(x) = 1$ if $x \in O_1$ and $f_1(x) = 0$ if $x \notin O_2$. Let $f_2 \in C(X)$ such that $0 \leq f_2(x) \leq 1$ for all $x \in X$, $f_2(x) = 1$ if $x \in O_{\delta/4}(\zeta)$ and $f_2(x) = 0$ if $x \notin O_{\delta/2}(\zeta)$. Then $f_1 f_2 = 0$. So $h_1(f_1)h_1(f_2) = 0$. Note that, by (e 3.10), $h_1(f_1)e_{n,i} = e_{n,i}$. So $h_1(f_2)e_{n,i} = 0$. If $y \in \Omega_{n,i}$ such that $s(y) = \zeta$. Then $h_1(f_2)(y)e_{n,i}(y) = f_2(s(y))e_{n,i}(y) \neq 0$. This is a contradiction. The claim is proved.

Now suppose that α_1 is minimal. To show that β_1 is minimal, let $\Omega_{n,i}$ be a clopen subset associated with the projection $e_{n,i}$. We will show that there exists an integer N such that $\beta_1^N(\xi) \in \Omega_{n,i}$. Let $Y_{n,i}^o$ be the interior of $Y_{n,i}$. Choose $x \in Y_{n,i}^o$ and $\epsilon > 0$ such that $\overline{O_\epsilon(x)} \subset Y_{n,i}^o$. Since α_1 is minimal, there is an integer N such that $\alpha_1^N(\zeta) \in O_{\epsilon/4}(x)$. There is $\delta > 0$ such that

$$\alpha_1^N(O_\delta(\zeta)) \subset O_{\epsilon/4}(x). \quad (\text{e 3.31})$$

Let d_n be as in the proof of 3.2. Choose m so that $d_m < \delta/32$. Note that, for some i' , $\xi \in \Omega_{m,i'}$. By the claim,

$$\overline{Y_{m,i'}} \cap O_{\delta/32}(\zeta) \neq \emptyset.$$

Therefore, since the diameter of $Y_{m,i'}$ is no more than d_m ,

$$\overline{Y_{m,i'}} \subset O_{\delta/8}(\zeta). \quad (\text{e 3.32})$$

Let $g \in C(X)$ such that $0 \leq g(x) \leq 1$ for all $t \in X$, $g(t) = 1$ if $t \in \overline{Y_{m,i'}}$ and $g(t) = 0$ if $t \notin O_{\delta/8}(\zeta)$. Let $g_1 \in C(X)$ such that $0 \leq g_1(t) \leq 1$, $g_1(t) = 1$ if $t \in O_{\epsilon/4}(x)$ and $g_1(t) = 0$ if $t \notin O_{\epsilon/2}(x)$.

Then (since $Y_{n,i}^o \supset \overline{O_\epsilon(x)}$), by (e 3.10),

$$h_1(g)e_{m,i'} = e_{m,i'} \text{ and } h_1(g_1)e_{n,i} = h_1(g_1).$$

Therefore (since $\beta_1^* \circ h_1 = h_1 \circ \alpha_1^*$)

$$h_1 \circ (\alpha_1^*)^N(g)(\beta_1^*)^N(e_{m,i'}) = (\beta_1^*)^N(e_{m,i'}). \quad (\text{e 3.33})$$

By (e 3.32) and (e 3.31),

$$h_1 \circ (\alpha_1^*)^N(g)h_1(g_1) = h_1 \circ (\alpha_1^*)^N(g). \quad (\text{e 3.34})$$

We also have, by applying (e 3.33) and (e 3.34),

$$\begin{aligned} e_{n,i}(\beta_1^*)^N(e_{m,i'}) &= e_{n,i}h_1 \circ (\alpha_1^*)^N(g)(\beta_1^*)^N(e_{m,i'}) \\ &= e_{n,i}h_1(g_1)h_1 \circ (\alpha_1^*)^N(g)(\beta_1^*)^N(e_{m,i'}) \\ &= h_1(g_1)h_1 \circ (\alpha_1^*)^N(g)(\beta_1^*)^N(e_{m,i'}) \\ &= h_1 \circ (\alpha_1^*)^N(g)(\beta_1^*)^N(e_{m,i'}) \\ &= (\beta_1^*)^N(e_{m,i'}) \end{aligned}$$

This implies that

$$\beta_1^N(\Omega_{m,i'}) \subset \Omega_{n,i}.$$

Hence

$$\beta_1^N(\xi) \in \Omega_{n,i}.$$

Therefore β_1 is minimal.

Since (Y, β_1) is minimal, Y can not have isolated points. It follows that Y is an infinite compact totally disconnected perfect metric space. Therefore Y is homeomorphic to a Cantor set.

Now suppose that (X, α_1) has a uniquely α_1 -invariant Borel probability measure. Note that μ induced by $\tau \circ h_2$ is a β_1 -invariant Borel probability measure on Y . Let μ_1 be another β_1 -invariant Borel probability measure on Y . Let τ_1 be the tracial state of $C(Y)$ defined by

$$\tau_1(g) = \int_X g d\mu_1 \text{ for } g \in C(Y).$$

Then $\tau_1 \circ h_1$ gives an α_1^* -invariant tracial state on $C(X)$. It follows that

$$\tau_1 \circ h_1(f) = \int_X f d\mu \text{ for } f \in C(X).$$

For any $Y_{n,i}$, by (e 3.10),

$$\tau_1(e_{n,i}) \geq \sup\left\{\int_X f d\mu : f \in C(X), 0 \leq f(x) \leq 1, f(x) = 0 \text{ if } x \notin Y_{n,i}^o\right\} = \mu(Y_{n,i}^o). \quad (\text{e 3.35})$$

Similarly,

$$\begin{aligned} \tau_1(1 - e_{n,i}) &\geq \sup\left\{\int_X f d\mu : f \in C(X), 0 \leq f(x) \leq 1, f(x) = 0 \text{ if } x \in \overline{Y_{n,i}}\right\} \\ &= \mu(X \setminus \overline{Y_{n,i}}). \end{aligned} \quad (\text{e 3.36})$$

Note, since $\mu(\partial(Y_{n,i})) = 0$,

$$\mu(X \setminus \overline{Y_{n,i}}) = \mu(X \setminus Y_{n,i}) \text{ and } \mu(Y_{n,i}) = \mu(Y_{n,i}^o).$$

We also have $\tau_1(e_{n,i}) + \tau_1(1 - e_{n,i}) = 1$ and $\mu(Y_{n,i}) + \mu(X \setminus Y_{n,i}) = 1$. Thus, by (e 3.35) and (e 3.36),

$$\tau_1(e_{n,i}) = \mu(Y_{n,i}).$$

We also have $\tau(e_{n,i}) = \mu(Y_{n,i})$. Since $\{e_{n,i}\}$ generates $C(Y)$, we conclude that

$$\tau_1 = \tau \circ h_2.$$

Thus Y has only one β_1 -invariant Borel probability measure. □

4 Unital simple AF-embedding

First we would like to point out that, in general, a C^* -algebra which can be embedded into a unital AF-algebra may not be embedded into a unital simple AF-algebra. A simplest example is to consider a unital AF-algebra which is defined by a unital essential extension:

$$0 \rightarrow \mathcal{K} \rightarrow E \rightarrow M_n \rightarrow 0$$

E can not be embedded into any unital simple AF-algebra since it does not admit a faithful tracial state.

Lemma 4.1. (Pasnicu-Phillips)

Let A be a unital C^* -algebra and let $\alpha \in \text{Aut}(A)$ be an automorphism which has the following version of cyclic Rokhlin property: for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any integer $n > 1$, there exist mutually orthogonal projections p_1, p_2, \dots, p_n in A such that

- (1) $\|ap_i - p_ia\| < \epsilon$ for all $a \in \mathcal{F}$ and $i = 1, 2, \dots, n$,
- (2) $\|\alpha(p_i) - p_{i+1}\| < \epsilon$ for $i = 1, 2, \dots, n-1$ and
- (3) $\sum_{i=1}^n p_i = 1$.

Then, for any $a \in A \rtimes_{\alpha} \mathbb{Z} \setminus \{0\}$ and any $\eta > 0$, there are mutually orthogonal projections $e_1, \dots, e_m \in A$ such that

$$\sum_{i=1}^m e_m = 1 \text{ and } \|E(a) - \sum_{i=1}^m e_i a e_i\| < \eta \quad (\text{e4.37})$$

where $E : A \rtimes_{\alpha} \mathbb{Z} \rightarrow A$ is the canonical conditional expectation. Consequently, if $I \subset A \rtimes_{\alpha} \mathbb{Z}$ is a proper ideal, then $I \cap A$ is a proper closed α -invariant ideal of A .

Corollary 4.2. In the situation in 4.1, if τ is a tracial state of $A \rtimes_{\alpha} \mathbb{Z}$, then there exists a α -invariant tracial state τ_1 of A such that

$$\tau(b) = \tau_1 \circ E(b) \text{ for all } b \in A \rtimes_{\alpha} \mathbb{Z}, \quad (\text{e4.38})$$

where $E : A \rtimes_{\alpha} \mathbb{Z} \rightarrow A$ is the standard conditional expectation.

Proof. Let $\epsilon > 0$. By 4.1, there are mutually orthogonal projections $e_1, \dots, e_m \in A$ such that

$$\sum_{i=1}^m e_m = 1 \text{ and } \|E(a) - \sum_{i=1}^m e_i a e_i\| < \epsilon \quad (\text{e4.39})$$

Let τ be a tracial state on $A \rtimes_{\alpha} \mathbb{Z}$. Then

$$\tau(e_i a e_i) = \tau(a e_i^2) = \tau(a e_i) \quad i = 1, 2, \dots, m. \quad (\text{e4.40})$$

Therefore

$$\tau\left(\sum_{i=1}^m e_i a e_i\right) = \sum_{i=1}^m \tau(e_i a e_i) \quad (\text{e4.41})$$

$$= \sum_{i=1}^m \tau(a e_i) = \tau\left(\sum_{i=1}^m a e_i\right) = \tau(a) \quad (\text{e4.42})$$

Let $\tau_1 = \tau|_A$, where we identify A with its image of natural embedding in $A \rtimes_{\alpha} \mathbb{Z}$. Then

$$|\tau(a) - \tau(E(a))| = \left| \tau\left(\sum_{i=1}^m e_i a e_i\right) - \tau_1 \circ E(a) \right| < \epsilon \quad (\text{e4.43})$$

for all $a \in A \rtimes_\alpha \mathbb{Z}$. It follows that

$$\tau = \tau_1 \circ E. \quad (\text{e4.44})$$

□

Theorem 4.3. *Let X be a compact metric space and let α be a homeomorphism. Then the following are equivalent.*

- (1) $C(X) \rtimes_\alpha \mathbb{Z}$ can be embedded into a unital simple AF-algebra.
- (2) There is a strictly positive α -invariant Borel probability measure on X .

Proof. Suppose that there is a monomorphism $\varphi : C(X) \rtimes_\alpha \mathbb{Z} \rightarrow C$, where C is a unital simple AF-algebra. Let τ be a tracial state of C . Then $\tau \circ \varphi$ gives a strictly positive α -invariant Borel probability measure. Thus (1) implies (2).

Now we apply 3.3 and a result of N. Brown to prove (2) \Rightarrow (1). It follows from the of 3.3 that there is a monomorphism $j_1 : C(X) \rightarrow C(Y)$, where Y is a compact totally disconnected space and a homeomorphism β on Y such that

$$\beta \circ j_1 = j_1 \circ \alpha.$$

Note that $C(Y) = C_{00}$ is an AF-algebra. Let C_0 be as in 3.2. Let \mathcal{U} and σ be as in 2.6. Denote by $j_2 : C(Y) \rightarrow C_0 \otimes \mathcal{U}$ the composition of the embeddings. Then

$$((\text{id}_{C_0} \otimes \sigma) \circ j_2)_* = (j_2 \circ \beta)_*. \quad (\text{e4.45})$$

Note that $\text{id}_{C_0} \otimes \sigma$ has the cyclic Rokhlin property of 2.6. It follows from 2.8 of [1] that there is a unitary $v \in C(Y)$, a unitary $u \in C_0 \otimes \mathcal{U}$ and a monomorphism $\varphi : C(Y) \rightarrow C_0 \otimes \mathcal{U}$ such that

$$(\text{ad } u \otimes \sigma) \circ \varphi = \varphi \circ \text{ad } v \circ \beta = \text{ad } \varphi(v) \circ \varphi \circ \beta. \quad (\text{e4.46})$$

Let $\gamma : C_0 \otimes \mathcal{U} \rightarrow C_0 \otimes \mathcal{U}$ be defined by $\text{ad } (\varphi(v^*)) \circ (u \otimes \sigma)$. Then $\gamma \circ \varphi = \varphi \circ \beta$. On $C_0 \otimes \mathcal{U} \otimes \mathcal{U}$ define $\tilde{\gamma} = \gamma \otimes \sigma$. We obtain an injective homomorphism

$$\tilde{\varphi} : (C(Y) \otimes \mathcal{U})_{\beta \otimes \sigma} \mathbb{Z} \rightarrow (C_0 \otimes \mathcal{U} \otimes \mathcal{U}) \rtimes_{\tilde{\gamma}} \mathbb{Z}. \quad (\text{e4.47})$$

Note that $(\tilde{\gamma})_{*0} = \text{id}_{K_0(C_0 \otimes \mathcal{U} \otimes \mathcal{U})}$. Since $\tilde{\gamma}$ has the cyclic Rokhlin property in 2.6, by 2.7, $C_0 \otimes \mathcal{U} \otimes \mathcal{U} \rtimes_{\tilde{\gamma}} \mathbb{Z}$ has tracial rank zero. Since it also satisfy the UCT, it is a unital simple AH-algebra with real rank zero and with no dimension growth. Since C_0 has a unique tracial state, so does $C_0 \otimes \mathcal{U} \otimes \mathcal{U}$. It follows from 4.2 that $C_0 \otimes \mathcal{U} \otimes \mathcal{U} \rtimes_{\tilde{\gamma}} \mathbb{Z}$ has a unique tracial state. Let \mathbb{D} be the tracial range of $K_0(C_0 \otimes \mathcal{U} \otimes \mathcal{U} \rtimes_{\tilde{\gamma}} \mathbb{Z})$. Let D be the unital simple AH-algebra with $(K_0(D), K_0(D)_+, [1_D]) = (\mathbb{D}, \mathbb{D}_+, 1)$. It follows that there is an injective homomorphism $\varphi_1 : C_0 \otimes \mathcal{U} \otimes \mathcal{U} \rtimes_{\tilde{\gamma}} \mathbb{Z} \rightarrow D$. We then first embed $C(X) \rtimes_\alpha \mathbb{Z}$ into $C(Y) \otimes \mathcal{U} \rtimes_{\beta \otimes \sigma} \mathbb{Z}$ and embed the latter into $C_0 \otimes \mathcal{U} \otimes \mathcal{U} \rtimes_{\tilde{\gamma}} \mathbb{Z}$. By composing this embedding with φ_1 , we obtain the desired embedding. □

Proposition 4.4. *Let A be a unital separable C^* -algebra. Then the following are equivalent*

- (1) A can be embedded into a unital simple AF-algebra;
- (2) A can be embedded into a unital simple AF-algebra with a unique tracial state;
- (3) A can be embedded into a unital separable amenable simple C^* -algebra with tracial rank zero which satisfies the UCT.

Proof. Suppose (1) holds. Let B be a unital simple AF-algebra and let $h : A \rightarrow B$ be an embedding. By replacing B by $h(1_A)Bh(1_A)$, we may assume that h is unital. Let $\tau \in T(B)$ be a tracial state. Define $r : K_0(B) \rightarrow \mathbb{R}$ by $r([e]) = \tau([e])$. Let

$$\mathbb{D} = \{r(x) : x \in K_0(B)\}. \quad (\text{e 4.48})$$

Then it is known that \mathbb{D} must be a countable dense subgroup of \mathbb{R} . Let D be a unital simple AF-algebra such that $(K_0(D), K_0(D)_+, [1_D]) = (\mathbb{D}, \mathbb{D}_+, 1)$. There is a unital monomorphism $h_1 : B \rightarrow D$. Thus $h_1 \circ h : A \rightarrow D$ gives an embedding of A into a unital simple AF-algebra with a unique tracial state.

That (2) \Rightarrow (3) is obvious.

Suppose that (3) holds. Let B_1 be a unital separable amenable simple C^* -algebra with tracial rank zero which satisfies the UCT and let $\varphi : A \rightarrow B_1$ is an embedding. Since $B_1 \otimes \mathcal{U}$ also satisfies the UCT, by the classification theorem ([9]), there is a monomorphism $\varphi_2 : B_1 \rightarrow C$ for some unital simple AF-algebra. This implies that (1) holds. \square

It should be noted that, when $d = 1$, Matui ([13]) proved that A can be embedded into an AF-algebra.

Theorem 4.5. *Let A be a unital separable simple amenable C^* -algebra with tracial rank zero and with a unique tracial state which satisfies the UCT. Suppose that $\Lambda : \mathbb{Z}^d \rightarrow \text{Aut}(A)$ is a homomorphism. Then $A \rtimes_{\Lambda} \mathbb{Z}^d$ can be embedded into a unital simple AF-algebra*

Proof. Suppose that Λ is determined by d mutually commuting automorphisms $\alpha_1, \alpha_2, \dots, \alpha_d$. Let \mathcal{U} and σ be as in 2.6. Put $A_1 = A \otimes \mathcal{U}$ and $\gamma_1 = \alpha_1 \otimes \sigma$. Define $B_1 = A_1 \rtimes_{\gamma_1} \mathbb{Z}$. Then γ_1 has the cyclic Rokhlin property (2.6). For any projection with the form $1_A \otimes e$, where $e \in \mathcal{U}$ is a projection, one has

$$[\gamma_1(1_A \otimes e)] = [1_A \otimes e] \text{ in } K_0(A_1).$$

Let $G_1 \subset K_0(A_1)$ be the subgroup generated by projections of the form $1_A \otimes e$. Then

$$(\gamma_1)_{*0}|_{G_1} = \text{id}_{G_1}.$$

Let τ be the unique tracial state on A_1 . Then

$$\{\tau(1_A \otimes e) : e \text{ projections in } M_k(\mathcal{U}), k = 1, 2, \dots\}$$

is dense in \mathbb{R} . Since A has a unique tracial state, so does A_1 . It follows that $\rho_{A_1}(G_1)$ is dense in $\text{Aff}(T(A_1))$. Then, by 2.7, $A_1 \rtimes_{\gamma_1} \mathbb{Z}$ has tracial rank zero and satisfies the UCT. We also note that there is an embedding:

$$A \rtimes_{\alpha_1} \mathbb{Z} \rightarrow (A \otimes \mathcal{U}) \rtimes_{\gamma_1} \mathbb{Z} = A_1 \rtimes \mathbb{Z}. \quad (\text{e 4.49})$$

In particular, if $d = 1$, by 4.4, the theorem follows.

If $d > 1$, put $B_1 = A_1 \rtimes_{\gamma_1} \mathbb{Z}$ and $A_2 = B_1 \otimes \mathcal{U}$. It follows from 4.2, since γ_1 has the cyclic Rokhlin property and A_1 has the unique tracial state, that B_1 has the tracial cyclic Rokhlin property. It follows that A_2 has a unique tracial state. Since $\alpha_j \otimes \text{id}_{\mathcal{U}}$ commutes with γ_1 , it gives an automorphism on B_1 . We denote it by $\alpha_{j,1}$. Define $\gamma_2 = \alpha_{2,1} \otimes \sigma$. Thus γ_2 is an automorphism on A_2 satisfying the cyclic Rokhlin property. Let $G_2 \subset K_0(A_2)$ generated by projections with the form $1_{B_1} \otimes e$, where $e \in \mathcal{U}$ is a projection. Then

$$\gamma_{2*0}|_{G_2} = \text{id}_{G_2}.$$

Since A_1 has a unique tracial state and γ_2 has the cyclic Rokhlin property, by applying 4.2, B_1 has a unique tracial state. We also have that $\rho_{A_2}(G_2)$ is dense in \mathbb{R} . It follows from 2.7 that $A_2 \rtimes_{\gamma_2} \mathbb{Z}$ has tracial rank zero and satisfies the UCT. Moreover, by (e 4.49), we have the following embedding

$$(A \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\alpha_2} \mathbb{Z} \rightarrow (B_1 \otimes \mathcal{U}) \rtimes_{\gamma_2} \mathbb{Z} = A_2 \rtimes_{\gamma_2} \mathbb{Z}. \quad (\text{e 4.50})$$

So, by 4.4, if $d = 2$, we obtain an embedding from $A \rtimes_{\Lambda} \mathbb{Z}^d$ into a unital simple AF-algebra.

If $d > 2$, put $B_2 = A_2 \rtimes_{\gamma_2} \mathbb{Z}$. We have shown that A_2 has a unique tracial state. Since γ_2 has the cyclic Rokhlin property, by 4.2, B_2 has a unique tracial state. It follows that $A_3 = B_2 \otimes \mathcal{U}$ has a unique tracial state. Note that $\alpha_{j,1} \otimes \text{id}_{\mathcal{U}}$ is an automorphism on A_2 . Since $\alpha_{j,1} \otimes \text{id}_{\mathcal{U}}$ commutes with γ_2 , it gives an automorphism $\alpha_{j,2}$ on B_2 . Define $\gamma_3 = \alpha_{3,2} \otimes \sigma$. It is an automorphism on A_3 satisfying the cyclic Rokhlin property. We then apply the same argument above to conclude that $A_3 \rtimes_{\gamma_3} \mathbb{Z}$ has tracial rank zero and satisfies the UCT. Moreover, there is an embedding:

$$((A \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\alpha_2} \mathbb{Z}) \rtimes_{\alpha_3} \mathbb{Z} \rightarrow (B_2 \otimes \mathcal{U}) \rtimes_{\gamma_3} \mathbb{Z} \quad (\text{e 4.51})$$

In particular, 4.4 implies that the theorem hold for $d = 3$. The theorem follows by applying the same argument and induction. \square

Corollary 4.6. *Let A be a unital separable simple amenable C^* -algebra with tracial rank zero and with a unique tracial state which satisfies the UCT and let G be a finitely generated discrete abelian group. Suppose that $\Lambda : G \rightarrow \text{Aut}(A)$ is a homomorphism. Then $A \rtimes_{\Lambda} G$ can be embedded into a unital simple AF-algebra*

Proof. We combine 4.5 with an argument of N. Brown. Write $G = \mathbb{Z}^d \oplus G_0$, where G_0 is a finite subgroup of G . Thus $G/\mathbb{Z}^d \cong G_0$ is compact. Put $B = A \rtimes_{\Lambda|_{\mathbb{Z}^d}} \mathbb{Z}^d$. By a theorem of Green (Cor. 2.8 of [5]),

$$A \otimes C(G_0) \rtimes_{\Lambda \otimes \Gamma} G \cong B \otimes M_k$$

for some integer $k \geq 1$. Now 4.5 asserts that $B \otimes M_k$ can be embedded into a unital simple AF-algebra. The corollary follows from the fact that there is a natural embedding $A \rtimes_{\Lambda} G \rightarrow A \otimes C(G_0) \rtimes_{\Lambda \otimes \Gamma} G$ (see also Remark 11.10 of [2]). \square

Now we consider crossed products $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$ for $d > 1$. It is easy to see that if $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$ can be embedded into a unital simple AF-algebra then there is a strictly positive Λ -invariant Borel probability measure. The following shows that, if in addition, (X, α_1) is minimal and unique ergodic for a generator α_1 of Λ , then the converse also holds.

Theorem 4.7. *Let X be a compact metric space and let (X, Λ) be a \mathbb{Z}^d system. Suppose that there exists a Λ -invariant strictly positive Borel probability measure. Suppose that, in addition, there is a generator α_1 of Λ for which (X, α_1) is minimal and unique ergodic. Then $C(X) \rtimes_{\Lambda} \mathbb{Z}^d$ can be embedded into a unital simple AF-algebra*

Proof. Suppose that X has infinitely many points. Let Y be the Cantor set. By 3.3 and 3.4, there is a covariant injective homomorphism $h_1 : C(X) \rtimes_{\Lambda} \mathbb{Z}^d \rightarrow C(Y) \rtimes_{\bar{\Lambda}} \mathbb{Z}^d$ such that (Y, β_1) is minimal and unique ergodic.

Therefore, it suffices to prove the theorem in the case that X is the Cantor set.

It follows from 2.1 of [17] that $C(X) \rtimes_{\alpha_1} \mathbb{Z}$ is a unital simple AT-algebra of real rank zero. Let $A = C(X) \rtimes_{\alpha_1} \mathbb{Z}$. Then $\alpha_2, \alpha_3, \dots, \alpha_d$ gives a \mathbb{Z}^{d-1} action Λ' on A . By the assumption that

(X, α_1) is unique ergodic, A has a unique tracial state. It follows from 4.5 that there is an embedding

$$j : A \rtimes_{\Lambda'} \mathbb{Z}^{d-1} \rightarrow C$$

for some unital simple AF-algebra C . But

$$A \rtimes_{\Lambda'} \mathbb{Z}^{d-1} \cong C(X) \rtimes_{\Lambda} \mathbb{Z}^d.$$

Thus we obtain an embedding

$$C(X) \rtimes_{\Lambda} \mathbb{Z}^d \cong A \rtimes_{\Lambda'} \mathbb{Z}^{d-1} \rightarrow C.$$

□

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